

Plane-parallel waves as duals of the flat background

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Abstract

We give a classification of non-Abelian T-duals of the flat metric in $D = 4$ dimensions with respect to the four-dimensional continuous subgroups of the Poincaré group. After dualizing the flat background, we identify majority of dual models as conformal sigma models in plane-parallel wave backgrounds, most of them having torsion. We give their form in Brinkmann coordinates. We find, besides the plane-parallel waves, several diagonalizable curved metrics with nontrivial scalar curvature and torsion. Using the non-Abelian T-duality, we find general solution of the classical field equations for all the sigma models in terms of d'Alembert solutions of the wave equation.

Contents

1	Introduction	2
2	Non-Abelian T-duality	4
3	Solving the classical sigma model equations by non-Abelian T-duality	6

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4	Strings in the pp-wave background	8
5	Examples	11
5.1	Example 1 – subalgebra S_{27}	11
5.1.1	Duals to the flat metric	11
5.1.2	Solution of the classical equations of the sigma model .	13
5.2	Example 2 – subalgebra S_{17}	17
5.3	Example 3 – subalgebra S_{19}	19
6	Results for other subalgebras	21
6.1	The pp-waves	22
6.1.1	Subalgebras S_7, S_8	22
6.1.2	Subalgebra S_{23}	24
6.1.3	Subalgebra S_{25}	25
6.1.4	Subalgebras S_{26}, S_{27}	26
6.1.5	Subalgebra S_{28}	28
6.1.6	Subalgebra S_{29}	29
6.1.7	Subalgebras S_{31}, S_{33}	30
6.2	Diagonalizable metrics with nontrivial scalar curvature	32
6.2.1	Subalgebra S_{11}	32
6.2.2	Subalgebra S_{18}	33
7	Conclusion	35

1 Introduction

String theory in curved and/or time-dependent background can be formulated as a sigma model satisfying supplementary conditions. Finding solutions of equations of motion in such backgrounds is usually very complicated. That is why every solvable case attracts considerable attention. An example of such a model is a string theory in the homogeneous plane-parallel wave background, solved in Ref. [1] in terms of Bessel functions. Plane-parallel (pp-)wave backgrounds in string theory have been repeatedly investigated in the past (see e.g. references in [2]). They not only give solvable models [3], but also allow one to study the behavior of strings near spacetime singularities [4, 5]. Moreover, to extract information about string behavior in a general curved background, one can take the Penrose limit [6], extended

to fields in string theory in [7], and study string behavior in the resulting plane-wave background.

Particular cases of four-dimensional pp-wave background in Rosen coordinates obtained from gauged WZW (Wess–Zumino–Witten) models were given in [8],[9], and [2], as

$$ds^2 = dudv + \frac{g_1(u')}{g_1(u')g_2(u) + q^2} dx_1^2 + \frac{g_2(u)}{g_1(u')g_2(u) + q^2} dx_2^2, \quad (1)$$

$$B_{12} = \frac{q}{g_1(u')g_2(u) + q^2},$$

where $u' = au + d$ ($a, d = \text{const}$) and the functions g_i can take any pair of the following values:

$$g(u) = 1, \quad u^2, \quad \tanh^2 u, \quad \tan^2 u, \quad u^{-2}, \quad \coth^2 u, \quad \cot^2 u. \quad (2)$$

It is mentioned in [8] that this background is dual to the flat space for $g_1 = 1$, $g_2 = u^2$. We shall show that several other cases of these backgrounds are dual to the flat space as well. Moreover, we shall use this fact for finding general solutions of classical sigma model field equations in these pp-wave backgrounds. We find, beside pp-waves, several curved backgrounds with diagonalizable metrics resembling black hole [10] and cosmological [11] solutions and we check that solutions obtained by non-Abelian T-duality satisfy sigma model field equations in these backgrounds as well.

We understand the non-Abelian T-duality [12] as a special case of Poisson–Lie T-duality [13] based on the structure of the Drinfeld double. For technical reasons we shall restrict to four spacetime dimensions, but the discussion can be extended to higher dimensions using the spectator fields or subgroups of Poincaré group in higher dimensions. Investigation of conformal invariance of pp-waves in higher dimensions can be found e.g. in [14],[15].

The plan of the paper is the following. In the next two sections we describe the method whereby the Poisson–Lie T-duality is used as a tool for the construction of dual models and their solution. In section 4, we review relevant properties concerning strings in the pp-wave background. Detailed discussion of particular examples is given in section 5. Section 6 summarizes results of dualization with respect to various subgroups of the Poincaré group. Subalgebras corresponding to these subgroups are listed in the appendix.

2 Non-Abelian T-duality

The sigma model on a manifold M is given by the classical action

$$\begin{aligned} S_F[X] &= - \int d\sigma_+ d\sigma_- (\partial_- X^\mu F_{\mu\nu}(X) \partial_+ X^\nu) = \\ &= \frac{1}{2} \int d\tau d\sigma [-\partial_\tau X^\mu G_{\mu\nu}(X) \partial_\tau X^\nu + \partial_\sigma X^\mu G_{\mu\nu}(X) \partial_\sigma X^\nu - 2\partial_\tau X^\mu B_{\mu\nu}(X) \partial_\sigma X^\nu], \end{aligned} \quad (3)$$

where F is a second order tensor field on M , with the metric and the NS-NS 2-form (torsion potential; NS standing for Neveu-Schwarz) given by the symmetric and antisymmetric part of F :

$$G_{\mu\nu} = \frac{1}{2}(F_{\mu\nu} + F_{\nu\mu}), \quad B_{\mu\nu} = \frac{1}{2}(F_{\mu\nu} - F_{\nu\mu}).$$

The worldsheet coordinates are

$$\sigma_+ = \frac{1}{\sqrt{2}}(\tau + \sigma), \quad \sigma_- = \frac{1}{\sqrt{2}}(\tau - \sigma).$$

The functions X^μ are determined by the composition $X^\mu(\tau, \sigma) = x^\mu(X(\tau, \sigma))$, where $X : \mathbb{R}^2 \ni (\tau, \sigma) \mapsto X(\tau, \sigma) \in M$ and $x^\mu : \mathbf{U}_p \rightarrow \mathbb{R}$ are components of a coordinate map on a neighborhood \mathbf{U}_p of an element $X(\tau, \sigma) = p \in M$.

The non-Abelian T-duality [12] of sigma models is a special case of Poisson-Lie T-duality [13],[16] that can be formulated by virtue of the Drinfeld double – a connected Lie group whose Lie algebra \mathfrak{d} can be decomposed into a pair of equally dimensional subalgebras $\mathfrak{g}, \tilde{\mathfrak{g}}$ that are maximally isotropic with respect to a symmetric ad-invariant non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on \mathfrak{d} .

The Drinfeld double suitable for non-Abelian T-duality is the semidirect product $G \ltimes \tilde{G}$, where the group G can be taken as a subgroup of the isometry group of the background, which, in our case, will be flat. The group \tilde{G} has to be chosen Abelian in order to satisfy the conditions of dualizability of the sigma model [13]. We shall focus on the case, when the isometry subgroup acts freely and transitively on the manifold, so that we can make the identification $G \approx M$. This is usually referred to as atomic duality. Let us summarize the main points of the construction of dual models.

Given the four-dimensional subgroup G of the isometry group generated by Killing vectors of the flat metric, the tensor F is symmetric and can be written as

$$F_{\mu\nu}(x) = G_{\mu\nu}(x) = e_\mu^a(g(x))(E_0)_{ab}e_\nu^b(g(x)), \quad (4)$$

where E_0 is a constant non-singular symmetric matrix, and $e_\mu^a(g(x))$ are components of the right-invariant forms $(dg)g^{-1}$ expressed in coordinates $\{x^\mu\}$ on the group G and the basis of its Lie algebra $\{T_a\}$.

Denoting the mutually dual bases of \mathfrak{g} and Abelian $\tilde{\mathfrak{g}}$ as $\{T_i\}$, $\{\tilde{T}^j\}$, we construct subspaces $\varepsilon^+ = \text{Span}(T_i + E_{0,ij}\tilde{T}^j)$, $\varepsilon^- = \text{Span}(T_i - E_{0,ji}\tilde{T}^j)$ that are orthogonal w.r.t. \langle, \rangle and span the whole Lie algebra \mathfrak{d} . The field equations for the sigma model on the group G can be rewritten as the equation

$$\langle (\partial_\pm l)l^{-1}, \varepsilon^\pm \rangle = 0, \quad (5)$$

for mapping l from the worldsheet in \mathbb{R}^2 into the Drinfeld double D .

Due to Drinfeld, there exists a unique decomposition (at least in the vicinity of the unit element of D) of an arbitrary element l of D as a product of elements from G and \tilde{G} . Solutions of equation (5) and solution of the equations of motion for the sigma model $X^\mu(\tau, \sigma) = x^\mu(g(\tau, \sigma))$ are related by

$$l(\tau, \sigma) = g(\tau, \sigma)\tilde{h}(\tau, \sigma) \in D,$$

where $\tilde{h}(\tau, \sigma) \in \tilde{G}$ fulfills the equations

$$\partial_\tau \tilde{h}_j = -v_j^\lambda G_{\lambda\nu} \partial_\sigma X^\nu, \quad (6)$$

$$\partial_\sigma \tilde{h}_j = -v_j^\lambda G_{\lambda\nu} \partial_\tau X^\nu, \quad (7)$$

with v_j^λ representing components of the left-invariant fields v_j on G in the group coordinates x^μ .

The metric and the torsion potential of the non-Abelian T-dual model can be obtained from the tensor \tilde{F} :

$$\tilde{F}_{\mu\nu}(\tilde{x}) = [E_0 + \tilde{\Pi}(\tilde{g}(\tilde{x}))]^{-1}, \quad (8)$$

where the matrix $\tilde{\Pi}$ is given by the adjoint representation of the Abelian subgroup \tilde{G} on the Lie algebra of the Drinfeld double in the mutually dual bases

$$\text{Ad}(\tilde{g})^T = \begin{pmatrix} \mathbf{1} & 0 \\ \tilde{\Pi}(\tilde{g}) & \mathbf{1} \end{pmatrix}.$$

The relation between the solution $X^\mu(\tau, \sigma)$ of the equations of motion of the sigma model given by F and the solution $\tilde{X}^\mu(\tau, \sigma) := \tilde{x}^\mu(\tilde{g}(\tau, \sigma))$ of the

sigma model given by \tilde{F} follows from two possible decompositions of elements l of the Drinfeld double:

$$g(\tau, \sigma)\tilde{h}(\tau, \sigma) = \tilde{g}(\tau, \sigma)h(\tau, \sigma), \quad (9)$$

where $g, h \in G$, $\tilde{g}, \tilde{h} \in \tilde{G}$. The map $\tilde{h} : \mathbb{R}^2 \rightarrow \tilde{G}$ that we need for this transformation is obtained from equations (6),(7).

3 Solving the classical sigma model equations by non-Abelian T-duality

Equation (9) defines the Poisson–Lie transformation between solutions of the equations of motion of the original sigma model and its dual. Its application may be rather complicated. To use it for finding the solution of the dual model, the following three steps must be achieved:

- Step 1: One has to know the solution $X^\mu(\tau, \sigma)$ of the sigma model given by $F_{\mu\nu}(x)$.
- Step 2: Given $X^\mu(\tau, \sigma)$, one has to find $\tilde{h}(\tau, \sigma)$, i.e. solve the system of PDEs (6),(7).
- Step 3: Given $l(\tau, \sigma) = g(\tau, \sigma)\tilde{h}(\tau, \sigma) \in D$, one has to find the dual decomposition $l(\tau, \sigma) = \tilde{g}(\tau, \sigma)h(\tau, \sigma)$, where $\tilde{g}(\tau, \sigma) \in \tilde{G}$, $h(\tau, \sigma) \in G$. Functions $\tilde{X}^\mu(\tau, \sigma) := \tilde{x}^\mu(\tilde{g}(\tau, \sigma))$ then solve the field equations of the dual sigma model.

For simplicity, we will restrict consideration to four spacetime dimensions. Our convention for the flat metric in the spacetime coordinates (t, x, y, z) is

$$\eta = \text{diag}(-1, 1, 1, 1).$$

It is easy to find solutions for the equations following from the flat metric in coordinates $x^I \in \{t, x, y, z\}$, as they reduce to two-dimensional wave equations

$$\partial_\tau^2 W^J - \partial_\sigma^2 W^J = 0, \quad J = t, x, y, z. \quad (10)$$

However, we need to identify the group G with the manifold, i.e. find appropriate coordinate transformation between (t, x, y, z) and the coordinates parametrizing the group. Choosing the parametrization of group elements as

$$g = g(x^\mu) = e^{x^1 T_1} e^{x^2 T_2} e^{x^3 T_3} e^{x^4 T_4}, \quad (11)$$

where T_j form the basis of the Lie algebra of the group, we may calculate the algebra of left-invariant fields

$$v_j = v_j^\mu \frac{\partial}{\partial x^\mu}, \quad j = 1, \dots, 4,$$

and compare it with the chosen four-dimensional subalgebra of Killing vectors (\mathcal{K}_i) of the flat metric in coordinates (t, x, y, z) . The comparison then may give the coordinate transformation

$$x^\mu = x^\mu(t, x, y, z)$$

as a solution to a set of PDEs.

The right-hand sides of the PDEs (6),(7), solved in step 2, are invariant w.r.t coordinate transformation. This means that we can express the right-hand sides in terms of the coordinates (t, x, y, z) and use the Killing fields \mathcal{K}_j instead of the left-invariant fields on G . The equations (6) and (7) then acquire the form

$$\partial_\tau \tilde{h}_j = -\mathcal{K}_j^I \eta_{IJ} \partial_\sigma W^J, \quad (12)$$

$$\partial_\sigma \tilde{h}_j = -\mathcal{K}_j^I \eta_{IJ} \partial_\tau W^J, \quad (13)$$

where the W^J are solutions of two-dimensional wave equations (10), and \mathcal{K}_j^I are the components of Killing vectors in coordinates (t, x, y, z) .

Step 3 represents in general rather complicated problem related to the Baker–Campbell–Hausdorff formula. Its solution simplifies substantially when the adjoint representation of the Lie algebra \mathfrak{g} is faithful. Let

$$l = g\tilde{h} = \tilde{g}h, \quad g, h \in G, \quad \tilde{g}, \tilde{h} \in \tilde{G} = \mathbb{R}^4, \quad (14)$$

and assume that the parametrizations of $g, h, \tilde{g}, \tilde{h}$ are

$$\begin{aligned} g &= e^{x^1 T_1} e^{x^2 T_2} e^{x^3 T_3} e^{x^4 T_4}, & \tilde{h} &= e^{\tilde{h}_1 \tilde{T}^1} e^{\tilde{h}_2 \tilde{T}^2} e^{\tilde{h}_3 \tilde{T}^3} e^{\tilde{h}_4 \tilde{T}^4}, \\ \tilde{g} &= e^{\tilde{x}_1 \tilde{T}^1} e^{\tilde{x}_2 \tilde{T}^2} e^{\tilde{x}_3 \tilde{T}^3} e^{\tilde{x}_4 \tilde{T}^4}, & h &= e^{h^1 T_1} e^{h^2 T_2} e^{h^3 T_3} e^{h^4 T_4}. \end{aligned}$$

The variables x^j, \tilde{h}_k and \tilde{x}_j, h^k represent two sets of coordinates in (the vicinity of the unit of) the Drinfeld double. To express \tilde{x}_j, h^k in terms of x^j, \tilde{h}_k , we can use a representation r of an element of the semi-Abelian Drinfeld double in the form of block matrices $(\dim \mathfrak{g} + 1) \times (\dim \mathfrak{g} + 1)$, such that

$$r(g) = \begin{pmatrix} Ad\,g & 0 \\ 0 & 1 \end{pmatrix}, \quad r(\tilde{h}) = \begin{pmatrix} \mathbf{1} & 0 \\ v(\tilde{h}) & 1 \end{pmatrix},$$

where $v(\tilde{h}) = (\tilde{h}_1, \dots, \tilde{h}_{\dim \mathfrak{g}})$. From the equation (14) we then get

$$r(l) = r(g\tilde{h}) = \begin{pmatrix} Ad g & 0 \\ v(\tilde{h}) & 1 \end{pmatrix} = r(\tilde{g}h) = \begin{pmatrix} Ad h & 0 \\ v(\tilde{g}) \cdot (Ad h) & 1 \end{pmatrix}. \quad (15)$$

If the adjoint representation of the Lie algebra \mathfrak{g} is faithful, then the representation r of the Drinfeld double is faithful as well, and the relation (15) gives a system of equations for \tilde{x}_j and h^j . If not, we can try to use formula

$$e^A e^B = e^{\exp(ad A)B} e^A \quad (16)$$

to permute the elements of G and \tilde{G} in (14) and express the coordinates \tilde{x}_j, h^k in terms of x^j, \tilde{h}_k .

In the following sections we shall apply the above given three steps of the Poisson–Lie transformation to solve the sigma model field equations in curved backgrounds dual to the flat metric.

4 Strings in the pp-wave background

We will be interested in the special subclass of metrics called pp-waves. Their metric in the so called Brinkmann coordinates $(u, v, z_3, z_4, \dots, z_D)$ can be written as

$$ds^2 = 2dudv - K(u, \vec{z})du^2 + d\vec{z}^2, \quad (17)$$

where $d\vec{z}^2$ is the Euclidean metric in the transversal space with coordinates $\vec{z} = (z_3, z_4, \dots, z_D)$. We denote the number of transversal coordinates by d , such that $D = 2 + d$. The NS–NS 2-form of particular interest to us has the form

$$B = B_j(u, \vec{z})du \wedge dz_j. \quad (18)$$

The metric (17) has covariantly constant null Killing vector ∂_v and particularly simple curvature properties, because the Ricci tensor has only one nonzero component

$$R_{uu} = \frac{1}{2}(\partial_3^2 K + \partial_4^2 K + \dots + \partial_D^2 K),$$

and the scalar curvature vanishes. The one-loop conformal invariance conditions for the sigma model

$$0 = R_{\mu\nu} - \nabla_\mu \nabla_\nu \Phi - \frac{1}{4} H_{\mu\kappa\lambda} H_\nu{}^{\kappa\lambda}, \quad (19)$$

$$0 = \nabla^\mu \Phi H_{\mu\kappa\lambda} + \nabla^\mu H_{\mu\kappa\lambda}, \quad (20)$$

$$0 = R - 2 \nabla_\mu \nabla^\mu \Phi - \nabla_\mu \Phi \nabla^\mu \Phi - \frac{1}{12} H_{\mu\kappa\lambda} H^{\mu\kappa\lambda}, \quad (21)$$

where $H = dB$, can be solved in some special cases. One of them is that of the model in the background resulting from the Penrose–Güven limit [6, 7], with

$$K(u, \vec{z}) = K_{ij}(u) z_i z_j, \quad (22)$$

and the torsion

$$H = H_{ij}(u) du \wedge dz_i \wedge dz_j \quad (23)$$

that follows from the NS–NS 2-form (18) if $B_j(u, \vec{z})$ is linear in z . The one-loop conformal invariance conditions then simplify to solvable differential equation for the dilaton $\Phi = \Phi(u)$:

$$\Phi''(u) - K_{jj}(u) + \frac{1}{4} H_{ij}(u) H_{ij}(u) = 0. \quad (24)$$

We are going to show that the sigma models in pp-wave backgrounds with special forms of the functions K_{ij} (22) and H_{ij} (23) can be obtained as non-Abelian T-duals of sigma models in the flat background. The Killing vectors of the flat metric $\eta = \text{diag}(-1, 1, 1, 1)$ in coordinates (t, x, y, z) are

$$P_0 = \partial_t, \quad P_j = \partial_j, \quad L_j = -\varepsilon_{ijk} x^j \partial_k, \quad M_j = -x^j \partial_t - t \partial_j, \quad (25)$$

and form the ten-dimensional Poincaré Lie algebra. To apply the (atomic) non-Abelian T-duality on sigma models in the flat background, we shall need four-dimensional subalgebras of the Poincaré Lie algebra, classified in [17].

For $K(u, \vec{z})$ in (17) at most quadratic in transversal coordinates, one can find transformations that bring it to the form (22). In the following, we will be able to bring the metrics of the resulting dual models with vanishing scalar curvature to the form (17), where

$$K(u, \vec{z}) = K_3(u) z_3^2 + K_4(u) z_4^2, \quad (26)$$

and the torsion is

$$H = H(u) du \wedge dz_3 \wedge dz_4.$$

Classical field equations of sigma model (3) in such a background then read

$$\partial_\tau^2 U - \partial_\sigma^2 U = 0, \quad (27)$$

$$\partial_\tau^2 Z_3 - \partial_\sigma^2 Z_3 = K_3(U) [(\partial_\sigma U)^2 - (\partial_\tau U)^2] Z_3 - H(U) [\partial_\sigma Z_4 \partial_\tau U - \partial_\tau Z_4 \partial_\sigma U], \quad (28)$$

$$\partial_\tau^2 Z_4 - \partial_\sigma^2 Z_4 = K_4(U) [(\partial_\sigma U)^2 - (\partial_\tau U)^2] Z_4 + H(U) [\partial_\sigma Z_3 \partial_\tau U - \partial_\tau Z_3 \partial_\sigma U], \quad (29)$$

$$\begin{aligned} \partial_\tau^2 V - \partial_\sigma^2 V = & H(U) [\partial_\sigma Z_4 \partial_\tau Z_3 - \partial_\sigma Z_3 \partial_\tau Z_4] + \\ & \sum_{j=3}^4 \left\{ 2K_j(U) Z_j [\partial_\tau Z_j \partial_\tau U - \partial_\sigma Z_j \partial_\sigma U] + \right. \\ & \left. (Z_j)^2 \left[\frac{1}{2} K'_j(U) [(\partial_\tau U)^2 - (\partial_\sigma U)^2] + K_j(U) (\partial_\tau^2 U - \partial_\sigma^2 U) \right] \right\}. \end{aligned} \quad (30)$$

For string backgrounds, the last equation can be replaced by the so-called string conditions for $X^\mu = (U, V, Z_3, Z_4)$

$$\partial_\tau X^\mu G_{\mu\nu}(X) \partial_\tau X^\nu + \partial_\sigma X^\mu G_{\mu\nu}(X) \partial_\sigma X^\nu = 0, \quad (31)$$

$$\partial_\tau X^\mu G_{\mu\nu}(X) \partial_\sigma X^\nu = 0. \quad (32)$$

Conditions (31),(32) for the pp-wave with function K given by (26) yield

$$2\partial_\tau U \partial_\tau V + \sum_{j=3}^4 \left\{ (\partial_\tau Z_j)^2 - (\partial_\tau U)^2 K_j(U) (Z_j)^2 \right\} + (\tau \rightarrow \sigma) = 0,$$

$$\partial_\tau U \partial_\sigma V + \partial_\tau V \partial_\sigma U + \sum_{j=3}^4 \left\{ \partial_\tau Z_j \partial_\sigma Z_j - \partial_\tau U \partial_\sigma U K_j(U) (Z_j)^2 \right\} = 0.$$

Compatibility of these two first order equations for $V = V(\tau, \sigma)$ is guaranteed by the equations (27) – (29).

Note that for nonvanishing torsion, both Z_3 and Z_4 appear in (28,29), so that even in the light-cone gauge $U = \kappa\tau$ these equations do not separate, and it can be rather difficult to solve them in the usual way using Fourier mode expansion. Nevertheless, the T-duality gives a method to obtain the general solution.

5 Examples

5.1 Example 1 – subalgebra S_{27}

We shall illustrate the above described methods of non-Abelian dualization of the flat metric on the example of Killing vectors

$$\begin{aligned}\mathcal{K}_1 &= M_3 = -z\partial_t - t\partial_z, \\ \mathcal{K}_2 &= L_2 + M_1 = -x\partial_t - (t+z)\partial_x + x\partial_z, \\ \mathcal{K}_3 &= L_1 - M_2 = y\partial_t + (t+z)\partial_y - y\partial_z, \\ \mathcal{K}_4 &= P_0 - P_3 = \partial_t - \partial_z\end{aligned}\tag{33}$$

that span subalgebra S_{27} (see the appendix). Their nonvanishing commutation relations are

$$[\mathcal{K}_1, \mathcal{K}_2] = -\mathcal{K}_2, \quad [\mathcal{K}_1, \mathcal{K}_3] = -\mathcal{K}_3, \quad [\mathcal{K}_1, \mathcal{K}_4] = -\mathcal{K}_4.\tag{34}$$

5.1.1 Duals to the flat metric

Using the parametrization (11) of the isometry subgroup G , where T_μ are elements of its Lie algebra commuting as in (34), we get the basis of left-invariant fields on G

$$\begin{aligned}v_1 &= \partial_1 + x^2\partial_2 + x^3\partial_3 + x^4\partial_4, \\ v_2 &= \partial_2, \quad v_3 = \partial_3, \quad v_4 = \partial_4.\end{aligned}$$

Identifying the Killing vectors (33) with these left-invariant fields, we get the transformation of coordinates on the flat manifold

$$\begin{aligned}t &= \frac{1}{2}e^{-x^1}((x^2)^2 + (x^3)^2 + 1) + x^4, & x &= -e^{-x^1}x^2, \\ z &= -\frac{1}{2}e^{-x^1}((x^2)^2 + (x^3)^2 - 1) - x^4, & y &= e^{-x^1}x^3\end{aligned}\tag{35}$$

that gives the flat metric in the group coordinates x^μ

$$G_{\mu\nu}(x) = F_{\mu\nu}(x) = \begin{pmatrix} 0 & 0 & 0 & e^{-x^1} \\ 0 & e^{-2x^1} & 0 & 0 \\ 0 & 0 & e^{-2x^1} & 0 \\ e^{-x^1} & 0 & 0 & 0 \end{pmatrix}.\tag{36}$$

This can be obtained from the formula (4) if one chooses

$$E_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The dual tensor \tilde{F} can be then found from the formula (8) as

$$\tilde{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{1-\tilde{x}_4} \\ 0 & 1 & 0 & \frac{\tilde{x}_2}{1-\tilde{x}_4} \\ 0 & 0 & 1 & \frac{\tilde{x}_3}{1-\tilde{x}_4} \\ \frac{1}{\tilde{x}_4+1} & -\frac{\tilde{x}_2}{\tilde{x}_4+1} & -\frac{\tilde{x}_3}{\tilde{x}_4+1} & \frac{\tilde{x}_2^2+\tilde{x}_3^2}{\tilde{x}_4^2-1} \end{pmatrix}.$$

Scalar curvature corresponding to the metric obtained from the symmetric part of this tensor vanishes, and Ricci tensor has only one nonvanishing component

$$\tilde{R}_{44} = -\frac{4}{(\tilde{x}_4^2 - 1)}.$$

This suggests that dual metric could be of the pp-wave form. Indeed, the transformation of coordinates for $|\tilde{x}_4| > 1$

$$\begin{aligned} \tilde{x}_1 &= v - \frac{1}{2}(z_3^2 + z_4^2) \coth(u), & \tilde{x}_2 &= z_3, \\ \tilde{x}_4 &= \coth(u), & \tilde{x}_3 &= z_4, \end{aligned} \quad (37)$$

brings the components of the tensor \tilde{F} into the form

$$\tilde{F} = \begin{pmatrix} 2 \frac{z_3^2 + z_4^2}{\sinh^2(u)} & 1 - \coth(u) & \frac{z_3}{\sinh^2(u)} & \frac{z_4}{\sinh^2(u)} \\ 1 + \coth(u) & 0 & 0 & 0 \\ -\frac{z_3}{\sinh^2(u)} & 0 & 1 & 0 \\ -\frac{z_4}{\sinh^2(u)} & 0 & 0 & 1 \end{pmatrix}. \quad (38)$$

The symmetric part yields pp-wave metric in the Brinkmann form

$$ds^2 = 2dudv + 2 \frac{z_3^2 + z_4^2}{\sinh^2(u)} du^2 + dz_3^2 + dz_4^2. \quad (39)$$

Torsion obtained from the antisymmetric part vanishes, and dilaton obtained as a solution of the equation (24) acquires a rather simple form

$$\Phi(u) = c_2 + c_1 u + 4 \log(\sinh(u)),$$

where c_1, c_2 are arbitrary constants.

For $|\tilde{x}_4| < 1$, the transformation

$$\begin{aligned} \tilde{x}_1 &= v - \frac{1}{2}(z_3^2 + z_4^2) \tanh(u), & \tilde{x}_2 &= z_3, \\ \tilde{x}_4 &= \tanh(u), & \tilde{x}_3 &= z_4, \end{aligned}$$

brings the tensor \tilde{F} into the form

$$\tilde{F} = \begin{pmatrix} -2 \frac{z_3^2 + z_4^2}{\cosh^2(u)} & 1 - \tanh(u) & -\frac{z_3}{\cosh^2(u)} & -\frac{z_4}{\cosh^2(u)} \\ 1 + \tanh(u) & 0 & 0 & 0 \\ \frac{z_3}{\cosh^2(u)} & 0 & 1 & 0 \\ \frac{z_4}{\cosh^2(u)} & 0 & 0 & 1 \end{pmatrix}, \quad (40)$$

giving the metric

$$ds^2 = 2dudv - 2 \frac{z_3^2 + z_4^2}{\cosh^2(u)} du^2 + dz_3^2 + dz_4^2. \quad (41)$$

Torsion again vanishes and dilaton has the form

$$\Phi(u) = c_2 + c_1 u + 4 \log(\cosh(u)). \quad (42)$$

We can see that the non-Abelian T-duality w.r.t. the subalgebra S_{27} produces two types of sigma models in the pp-wave backgrounds, one of them singular and one regular. As we shall see, this result is obtained also from dualization w.r.t. several other subalgebras of the Poincaré algebra.

5.1.2 Solution of the classical equations of the sigma model

Our next goal is to write down the general solution of the classical field equations in the backgrounds (38) and (40). As their torsions vanish, the antisymmetric parts do not contribute to the classical field equations.

The Lagrangian for the metric (39) can be written in the form (cf. (3))

$$L = \left[\frac{Z_3^2 + Z_4^2}{\sinh^2(U)} (\partial_\sigma U)^2 + \partial_\sigma U \partial_\sigma V + \frac{1}{2} (\partial_\sigma Z_3)^2 + \frac{1}{2} (\partial_\sigma Z_4)^2 \right] \\ - \left[\frac{Z_3^2 + Z_4^2}{\sinh^2(U)} (\partial_\tau U)^2 + \partial_\tau U \partial_\tau V + \frac{1}{2} (\partial_\tau Z_3)^2 + \frac{1}{2} (\partial_\tau Z_4)^2 \right].$$

The field equations then read

$$\partial_\tau^2 U - \partial_\sigma^2 U = 0, \quad (43)$$

$$\partial_\sigma^2 Z_3 - \partial_\tau^2 Z_3 = 2 \left((\partial_\sigma U)^2 - (\partial_\tau U)^2 \right) \frac{Z_3}{\sinh^2(U)}, \quad (44)$$

$$\partial_\sigma^2 Z_4 - \partial_\tau^2 Z_4 = 2 \left((\partial_\sigma U)^2 - (\partial_\tau U)^2 \right) \frac{Z_4}{\sinh^2(U)}, \quad (45)$$

$$\begin{aligned} \partial_\tau^2 V - \partial_\sigma^2 V = & 4 \operatorname{csch}^2(U) [Z_3 (\partial_\sigma U \partial_\sigma Z_3 - \partial_\tau U \partial_\tau Z_3) \\ & + Z_4 (\partial_\sigma U \partial_\sigma Z_4 - \partial_\tau U \partial_\tau Z_4)] \\ & - 2 \operatorname{csch}^3(U) [(Z_3)^2 + (Z_4)^2] [(-\partial_\sigma^2 U + \partial_\tau^2 U) \sinh(U) \\ & + ((\partial_\sigma U)^2 - (\partial_\tau U)^2) \cosh(U)]. \end{aligned} \quad (46)$$

To solve these field equations, we can follow steps 1–3 from the section 2.

Step 1 starts with solution of the field equations in the flat background. In the coordinates (t, x, y, z) they are of the form (10) solved by

$$W^I(\tau, \sigma) = R^I(\tau - \sigma) + L^I(\tau + \sigma), \quad I = t, x, y, z,$$

with R^I, L^I arbitrary functions. Subsequent transformation of this solution to the coordinates x^μ by using formulas (35) produces the functions

$$\begin{aligned} X^1(\tau, \sigma) &= -\log(W^t + W^z), & X^2(\tau, \sigma) &= -\frac{W^x}{W^t + W^z}, \\ X^4(\tau, \sigma) &= \frac{(W^t)^2 - (W^x)^2 - (W^y)^2 - (W^z)^2}{2(W^t + W^z)}, & X^3(\tau, \sigma) &= \frac{W^y}{W^t + W^z} \end{aligned}$$

that solve the sigma model field equations in the flat background in the coordinates x^μ , i.e. in metric (36).

Next, we have to perform step 2, consisting in the solution of the PDEs (12), (13), with Killing fields (33) on the right-hand sides. The equations (12) in this case read

$$\begin{aligned}\partial_\tau \tilde{h}_1 &= W^t \partial_\sigma W^z - W^z \partial_\sigma W^t, \\ \partial_\tau \tilde{h}_2 &= -W^x (\partial_\sigma W^t + \partial_\sigma W^z) + (W^t + W^z) \partial_\sigma W^x, \\ \partial_\tau \tilde{h}_3 &= W^y (\partial_\sigma W^t + \partial_\sigma W^z) - (W^t + W^z) \partial_\sigma W^y, \\ \partial_\tau \tilde{h}_4 &= \partial_\sigma W^t + \partial_\sigma W^z,\end{aligned}$$

while equations (13) are obtained by making the exchange $\tau \leftrightarrow \sigma$. Compatibility of these two sets of PDEs is guaranteed by the wave equations for W^I . Their solution is

$$\begin{aligned}\tilde{h}_1(\tau, \sigma) &= \gamma_1 + \int (W^t \partial_\sigma W^z - W^z \partial_\sigma W^t) d\tau, \\ \tilde{h}_2(\tau, \sigma) &= \gamma_2 - \int (W^x (\partial_\sigma W^t + \partial_\sigma W^z) - (W^t + W^z) \partial_\sigma W^x) d\tau, \\ \tilde{h}_3(\tau, \sigma) &= \gamma_3 + \int (W^y (\partial_\sigma W^t + \partial_\sigma W^z) - (W^t + W^z) \partial_\sigma W^y) d\tau, \\ \tilde{h}_4(\tau, \sigma) &= \gamma_4 + \int (\partial_\sigma W^t + \partial_\sigma W^z) d\tau,\end{aligned}\quad (47)$$

where $\gamma_1, \dots, \gamma_4$ are constants.

To get the solution of field equations (43)–(46) we have to carry out step 3. One can easily check that the adjoint representation of the algebra (34) is faithful, so we can use equation (15) to express the coordinates \tilde{x}_μ in terms of x^ν and \tilde{h}_k . We get

$$\begin{aligned}\tilde{x}_1 &= \tilde{h}_1 - x^2 \tilde{h}_2 - x^3 \tilde{h}_3 - x^4 \tilde{h}_4, \\ \tilde{x}_2 &= e^{x^1} \tilde{h}_2, \quad \tilde{x}_3 = e^{x^1} \tilde{h}_3, \quad \tilde{x}_4 = e^{x^1} \tilde{h}_4.\end{aligned}\quad (48)$$

Finally, we have to transform the coordinates \tilde{x}_μ into the Brinkmann's. Composing the inverse of (35), (48) and the inverse of (37), we get the Brinkmann coordinates (u, v, z_3, z_4) on \tilde{G} as functions of the spacetime coordinates (t, x, y, z) on the initial flat manifold and coordinates \tilde{h}_j on the subgroup \tilde{G} of the Drinfeld double

$$u = \operatorname{arccoth} \left(\frac{\tilde{h}_4}{t + z} \right), \quad z_3 = \frac{\tilde{h}_2}{t + z}, \quad z_4 = \frac{\tilde{h}_3}{t + z}, \quad (49)$$

$$v = \frac{\left(2\tilde{h}_2x - 2\tilde{h}_3y + 2\tilde{h}_1(t+z) + \tilde{h}_4(-t^2 + x^2 + y^2 + z^2)\right)}{2(t+z)} + \frac{\tilde{h}_4\tilde{h}_2^2 + \tilde{h}_3^2\tilde{h}_4}{2(t+z)^3}.$$

To get the general solution of the the classical field equations (43)–(46) in the curved background with the metric (39), we have to replace the coordinates (t, x, y, z) in (49) by the solutions $W^I = W^I(\tau, \sigma)$ of the wave equations (10) and \tilde{h}_μ by the solutions (47) of the PDEs (12), (13). We obtain

$$U(\tau, \sigma) = \operatorname{arccoth} \left(\frac{\tilde{h}_4(\tau, \sigma)}{W^t(\tau, \sigma) + W^z(\tau, \sigma)} \right), \quad (50)$$

$$Z_3(\tau, \sigma) = \frac{\tilde{h}_2(\tau, \sigma)}{W^t(\tau, \sigma) + W^z(\tau, \sigma)}, \quad Z_4(\tau, \sigma) = \frac{\tilde{h}_3(\tau, \sigma)}{W^t(\tau, \sigma) + W^z(\tau, \sigma)}.$$

The expression for the function $V(\tau, \sigma)$ is rather extensive, but can be easily read out of (49).

String-type solutions in the light-cone gauge (see e.g. [1],[18]), i.e.

$$U(\tau, \sigma) = \kappa\tau, \quad Z_3(\tau, \sigma) = \sum_{n=-\infty}^{\infty} Z_3^n(\tau) e^{2in\sigma}, \quad Z_4(\tau, \sigma) = \sum_{n=-\infty}^{\infty} Z_4^n(\tau) e^{2in\sigma}, \quad (51)$$

are obtained if

$$\begin{aligned} W^t(\tau, \sigma) + W^z(\tau, \sigma) &= e^{\kappa\sigma} \sinh(\kappa\tau), \\ W^x(\tau, \sigma) &= \sinh(\kappa\tau) \sum_{n=-\infty}^{\infty} e^{2in\sigma} (2in + \kappa) \int Z_3^n(\tau) \operatorname{csch}(\kappa\tau) d\tau, \\ W^y(\tau, \sigma) &= \sinh(\kappa\tau) \sum_{n=-\infty}^{\infty} e^{2in\sigma} (2in + \kappa) \int Z_4^n(\tau) \operatorname{csch}(\kappa\tau) d\tau, \end{aligned}$$

where $Z_3^n(\tau)$ and $Z_4^n(\tau)$ solve the differential equation

$$Z''(\tau) + (4n^2 - 2\kappa^2 \operatorname{csch}^2(\kappa\tau)) Z(\tau) = 0.$$

The solution of the classical field equations in the curved background with the metric (41) is obtained from the solution (50) when $\operatorname{arccoth}$ is replaced by $\operatorname{arctanh}$.

5.2 Example 2 – subalgebra S_{17}

The second example will deal with the subalgebra

$$S_{17} = \text{Span}[\mathcal{K}_1 = L_3 + \epsilon(P_0 + P_3), \mathcal{K}_2 = P_1, \mathcal{K}_3 = P_2, \mathcal{K}_4 = P_0 - P_3], \quad \epsilon = \pm 1$$

which produces a dual model with torsion and whose representation is not faithful. The commutation relations of this subalgebra are

$$[\mathcal{K}_1, \mathcal{K}_2] = \mathcal{K}_3, \quad [\mathcal{K}_1, \mathcal{K}_3] = -\mathcal{K}_2.$$

Transformation of coordinates in the flat background

$$t = x^1 \epsilon + x^4, \quad x = x^2, \quad y = x^3, \quad z = x^1 \epsilon - x^4, \quad (52)$$

yields components of the flat metric in the group coordinates as

$$F_{\mu\nu}(x) = \begin{pmatrix} 0 & 0 & 0 & -2\epsilon \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2\epsilon & 0 & 0 & 0 \end{pmatrix}.$$

The dual background in this case is

$$\tilde{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2\epsilon} \\ 0 & 1 & 0 & \frac{\tilde{x}_3}{2\epsilon} \\ 0 & 0 & 1 & -\frac{\tilde{x}_2}{2\epsilon} \\ -\frac{1}{2\epsilon} & -\frac{\tilde{x}_3}{2\epsilon} & \frac{\tilde{x}_2}{2\epsilon} & -\frac{\tilde{x}_2^2 + \tilde{x}_3^2}{4\epsilon^2} \end{pmatrix},$$

and the transformation to Brinkmann coordinates

$$\tilde{x}_1 = -v, \quad \tilde{x}_2 = z_3, \quad \tilde{x}_3 = z_4, \quad \tilde{x}_4 = 2\epsilon u, \quad (53)$$

brings the dual metric into the homogeneous and isotropic form

$$ds^2 = 2dudv - (z_3^2 + z_4^2) du^2 + dz_3^2 + dz_4^2. \quad (54)$$

The torsion in Brinkmann coordinates is constant

$$H = -2 du \wedge dz_3 \wedge dz_4, \quad (55)$$

and the dilaton is

$$\Phi(u) = c_1 + c_2 u.$$

To find the general solution of the field equations of the dual sigma model with torsion, we have to express the coordinates \tilde{x}_μ in terms of x^ν and \tilde{h}_k . As the adjoint representation of S_{17} is not faithful, we have to use the formula (16) to solve the equation (9) for coordinates of \tilde{g} . We get

$$\begin{aligned}\tilde{x}_1 &= \tilde{h}_1 + x^2 \tilde{h}_3 - x^3 \tilde{h}_2, & \tilde{x}_2 &= \tilde{h}_2 \cos x^1 - \tilde{h}_3 \sin x^1, \\ \tilde{x}_4 &= \tilde{h}_4, & \tilde{x}_3 &= \tilde{h}_2 \sin x^1 + \tilde{h}_3 \cos x^1.\end{aligned}$$

Like in the previous section, combining this with (53) and (52), we find the general solution of the field equations of the sigma model with metric (54) and torsion (55) as

$$\begin{aligned}U(\tau, \sigma) &= \frac{\tilde{h}_4(\tau, \sigma)}{2\epsilon}, \\ V(\tau, \sigma) &= -\tilde{h}_1(\tau, \sigma) - \tilde{h}_3(\tau, \sigma)W^x(\tau, \sigma) + \tilde{h}_2(\tau, \sigma)W^y(\tau, \sigma), \\ Z_3(\tau, \sigma) &= \cos(\Omega(\tau, \sigma))\tilde{h}_2(\tau, \sigma) - \sin(\Omega(\tau, \sigma))\tilde{h}_3(\tau, \sigma), \\ Z_4(\tau, \sigma) &= \cos(\Omega(\tau, \sigma))\tilde{h}_3(\tau, \sigma) + \sin(\Omega(\tau, \sigma))\tilde{h}_2(\tau, \sigma),\end{aligned}$$

where the $W^I(\tau, \sigma)$ are solutions of the wave equations (10),

$$\Omega(\tau, \sigma) = \frac{W^t + W^z}{2\epsilon},$$

and \tilde{h}_μ are solutions of the PDEs (12), (13),

$$\begin{aligned}\tilde{h}_1 &= \gamma_1 + \int \left[\epsilon (\partial_\tau W^t - \partial_\tau W^z) + W^x \partial_\tau W^y - W^y \partial_\tau W^x \right] d\sigma, \\ \tilde{h}_2 &= \gamma_2 - \int \partial_\tau W^x d\sigma, \quad \tilde{h}_3 = \gamma_3 - \int \partial_\tau W^y d\sigma, \\ \tilde{h}_4 &= \gamma_4 + \int (\partial_\tau W^t + \partial_\tau W^z) d\sigma.\end{aligned}$$

String-type solutions in the light-cone gauge (51) are obtained if we choose

$$W^t(\tau, \sigma) + W^z(\tau, \sigma) = 2\epsilon\kappa\sigma,$$

$$\begin{aligned}W^x(\tau, \sigma) &= \sum_{n=-\infty}^{\infty} e^{2in\sigma} \int Z_3^n(\tau) (\kappa \sin(\kappa\sigma) - 2in \cos(\kappa\sigma)) \\ &\quad - Z_4^n(\tau) (\kappa \cos(\kappa\sigma) + 2in \sin(\kappa\sigma)) d\tau,\end{aligned}$$

$$W^y(\tau, \sigma) = \sum_{n=-\infty}^{\infty} e^{2in\sigma} \int Z_3^n(\tau) (\kappa \cos(\kappa\sigma) + 2i n \sin(\kappa\sigma)) \\ + Z_4^n(\tau) (\kappa \sin(\kappa\sigma) - 2i n \cos(\kappa\sigma)) d\tau.$$

where $Z_3^n(\tau)$ and $Z_4^n(\tau)$ solve the system of differential equations

$$Z_3^{n''}(\tau) + (4n^2 + \kappa^2) Z_3^n(\tau) - 4i n \kappa Z_4^n(\tau) = 0, \\ Z_4^{n''}(\tau) + (4n^2 + \kappa^2) Z_4^n(\tau) + 4i n \kappa Z_3^n(\tau) = 0.$$

5.3 Example 3 – subalgebra S_{19}

The third example will deal with the subalgebra

$$S_{19} = \text{Span}[\mathcal{K}_1 = L_3 + \alpha P_3, \mathcal{K}_2 = P_1, \mathcal{K}_3 = P_2, \mathcal{K}_4 = P_0], \quad \alpha \neq 0$$

which produces diagonalizable dual metric with nonvanishing scalar curvature and torsion.

The commutation relations of this subalgebra

$$[\mathcal{K}_1, \mathcal{K}_2] = \mathcal{K}_3, \quad [\mathcal{K}_1, \mathcal{K}_3] = -\mathcal{K}_2$$

are equal to those in the previous example, but the subalgebras of Killing vectors cannot be transformed into one another by an element of the group of proper orthochronous Poincaré transformations (see [17]), and the representations of the commutation relations in Killing vector fields on M are different. This leads to a different transformation of coordinates in the flat background, namely,

$$x^1 = \frac{z}{\alpha}, \quad x^2 = x, \quad x^3 = y, \quad x^4 = t. \quad (56)$$

The components of the flat metric in the group coordinates then read

$$F_{\mu\nu} = \begin{pmatrix} \alpha^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The dual background in this case is

$$\tilde{F}_{\mu\nu} = \begin{pmatrix} \frac{1}{\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2} & \frac{\tilde{x}_3}{\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2} & -\frac{\tilde{x}_2}{\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2} & 0 \\ -\frac{\tilde{x}_3}{\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2} & \frac{\alpha^2 + \tilde{x}_2^2}{\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2} & \frac{\tilde{x}_2 \tilde{x}_3}{\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2} & 0 \\ \frac{\tilde{x}_2}{\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2} & \frac{\tilde{x}_2 \tilde{x}_3}{\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2} & \frac{\alpha^2 + \tilde{x}_3^2}{\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and its symmetric part gives a metric with nonvanishing scalar curvature

$$\tilde{R} = -\frac{4(\tilde{x}_2^2 + \tilde{x}_3^2) - 10\alpha^2}{(\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2)^2}.$$

This means that it cannot be transformed to the pp-wave form. On the other hand, the metric of this background can be diagonalized to the form

$$ds^2 = -dy_1^2 + dy_2^2 + \frac{y_2^2 \alpha^2}{y_2^2 + \alpha^2} dy_3^2 + \frac{1}{y_2^2 + \alpha^2} dy_4^2, \quad (57)$$

via

$$\tilde{x}_1 = y_4, \quad \tilde{x}_2 = y_2 \cos y_3, \quad \tilde{x}_3 = y_2 \sin y_3, \quad \tilde{x}_4 = y_1. \quad (58)$$

The torsion then acquires the form

$$H = \frac{2y_2 \alpha^2}{(y_2^2 + \alpha^2)^2} dy_2 \wedge dy_3 \wedge dy_4, \quad (59)$$

and the dilaton satisfying (19)–(21) is

$$\Phi = \log(y_2^2 + \alpha^2) + \text{const.}$$

To find the general solution of the field equations of this dual sigma model, we have to express the coordinates \tilde{x}_μ in terms of x^ν and \tilde{h}_k . As the adjoint representation of S_{19} is not faithful, we have to use the formula (16) to solve the equation (9) for the coordinates of \tilde{g} . We get

$$\begin{aligned} \tilde{x}_1 &= \tilde{h}_1 + x^2 \tilde{h}_3 - x^3 \tilde{h}_2, & \tilde{x}_2 &= \tilde{h}_2 \cos x^1 - \tilde{h}_3 \sin x^1, \\ \tilde{x}_4 &= \tilde{h}_4, & \tilde{x}_3 &= \tilde{h}_2 \sin x^1 + \tilde{h}_3 \cos x^1. \end{aligned}$$

Like in the previous section, combining this with (58) and (56), we find general solution of the field equations of the sigma model with metric (57) and torsion (59) as

$$\begin{aligned}
Y_1(\tau, \sigma) &= \tilde{h}_4(\tau, \sigma), \\
Y_2(\tau, \sigma) &= \sqrt{\tilde{h}_2(\tau, \sigma)^2 + \tilde{h}_3(\tau, \sigma)^2}, \\
Y_3(\tau, \sigma) &= \arctan \left(\frac{\cos(\Omega(\tau, \sigma))\tilde{h}_3(\tau, \sigma) + \sin(\Omega(\tau, \sigma))\tilde{h}_2(\tau, \sigma)}{\cos(\Omega(\tau, \sigma))\tilde{h}_2(\tau, \sigma) - \sin(\Omega(\tau, \sigma))\tilde{h}_3(\tau, \sigma)} \right), \\
Y_4(\tau, \sigma) &= \tilde{h}_1(\tau, \sigma) + \tilde{h}_3(\tau, \sigma)W^x(\tau, \sigma) - \tilde{h}_2(\tau, \sigma)W^y(\tau, \sigma),
\end{aligned}$$

where $W^I(\tau, \sigma)$ are solutions of the wave equations (10), $\Omega(\tau, \sigma) = \frac{W^z(\tau, \sigma)}{\alpha}$, and the \tilde{h}_μ are solutions of the PDEs (12),(13),

$$\begin{aligned}
\tilde{h}_1 &= \gamma_1 - \int \left[\alpha \partial_\tau W^z + W^y \partial_\tau W^x - W^x \partial_\tau W^y \right] d\sigma, \\
\tilde{h}_2 &= \gamma_2 - \int \partial_\tau W^x d\sigma, \quad \tilde{h}_3 = \gamma_3 - \int \partial_\tau W^y d\sigma, \\
\tilde{h}_4 &= \gamma_4 + \int \partial_\tau W^t d\sigma.
\end{aligned}$$

As this background is not of the pp-wave form, the light-cone gauge cannot be implemented [4]. Nevertheless, the field equations are solvable.

6 Results for other subalgebras

The classification of subalgebras of the Poincaré algebra in [17] was carried out up to the group of inner automorphisms of the connected component of the Poincaré group (proper orthochronous Poincaré transformations). There are 35 inequivalent four-dimensional subalgebras of the Poincaré algebra generated by Killing vectors (25).

Only the subgroups corresponding to the subalgebras $S_1, S_2, S_6, S_7, S_8, S_{11}, S_{17}, S_{18}, S_{19}, S_{23}, S_{25} - S_{29}, S_{31}, S_{33}$, listed in the appendix, act transitively and freely on the flat spacetime and can be used for the atomic non-Abelian T-duality. Non-Abelian duals generated by the algebras S_1, S_2, S_6 give backgrounds with flat metric and vanishing torsion. We will not discuss them further. Dual backgrounds obtained from duality w.r.t. the subalgebras S_{11} ,

S_{18} , S_{19} have nontrivial scalar curvature. The others are pp-waves, most of them with nonzero torsion, as we shall see from the following list of results. We do not repeat results for subalgebras S_{27} , S_{17} , S_{19} described in section 5.

6.1 The pp-waves

6.1.1 Subalgebras S_7, S_8

The non-isomorphic subalgebras

$$\begin{aligned} S_7 &= \text{Span}[\mathcal{K}_1 = 2M_3 + \alpha P_1, \mathcal{K}_2 = L_2 + M_1, \mathcal{K}_3 = P_0 - P_3, \mathcal{K}_4 = P_2], \\ S_8 &= \text{Span}[\mathcal{K}_1 = M_3, \mathcal{K}_2 = L_2 + M_1, \mathcal{K}_3 = P_0 - P_3, \mathcal{K}_4 = P_2], \end{aligned}$$

differ only in the value of the parameter α that is positive for S_7 , while $\alpha = 0$ for S_8 [17]. The commutation relations of S_7 are

$$[\mathcal{K}_1, \mathcal{K}_2] = -2\mathcal{K}_2 - \alpha\mathcal{K}_3, \quad [\mathcal{K}_1, \mathcal{K}_3] = -2\mathcal{K}_3.$$

The transformation of coordinates in the flat background

$$\begin{aligned} t &= x^1 x^2 (-\alpha) + \frac{1}{2} e^{-2x^1} ((x^2)^2 + 1) + x^3, & x &= x^1 \alpha - e^{-2x^1} x^2, \\ z &= x^1 x^2 \alpha - \frac{1}{2} e^{-2x^1} ((x^2)^2 - 1) - x^3, & y &= x^4, \end{aligned}$$

gives components of the flat metric in the group coordinates

$$F_{\mu\nu}(x) = \begin{pmatrix} \alpha^2 & -e^{-2x^1} \alpha(2x^1 + 1) & 2e^{-2x^1} & 0 \\ -e^{-2x^1} \alpha(2x^1 + 1) & e^{-4x^1} & 0 & 0 \\ 2e^{-2x^1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The dual background in this case is

$$\tilde{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix} 0 & 0 & \frac{1}{2-2\tilde{x}_3} & 0 \\ 0 & 1 & \frac{\tilde{x}_3 \alpha + \alpha + 2\tilde{x}_2}{2-2\tilde{x}_3} & 0 \\ \frac{1}{2\tilde{x}_3+2} & \frac{-\tilde{x}_3 \alpha + \alpha - 2\tilde{x}_2}{2\tilde{x}_3+2} & \frac{(2\tilde{x}_2 + \alpha \tilde{x}_3)^2}{4(\tilde{x}_3^2 - 1)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and the torsion vanishes.

The transformation to Brinkmann coordinates valid for $|\tilde{x}_3| < 1$,

$$\begin{aligned}\tilde{x}_1 &= -2v - \frac{1}{4} \left(\alpha^2 u + 4z_3 \alpha + 2z_3 \alpha \log(1 - \tanh^2(u)) \right) \\ &+ \frac{1}{16} \left[\tanh(u) \left[\alpha^2 \log^2(1 - \tanh^2(u)) + 4\alpha^2 \log(1 - \tanh^2(u)) + 16z_3^2 + 4\alpha^2 \right] \right. \\ \tilde{x}_2 &= z_3 - \frac{1}{4} \alpha \tanh(u) \log(1 - \tanh^2(u)), \\ \tilde{x}_3 &= -\tanh(u), \quad \tilde{x}_4 = z_4.\end{aligned}\tag{60}$$

brings the dual metric and dilaton to forms

$$ds^2 = 2dudv - 2 \frac{z_3^2}{\cosh^2(u)} du^2 + dz_3^2 + dz_4^2,\tag{61}$$

$$\Phi(u) = c_1 + c_2 u + 2 \log(\cosh(u)).$$

The transformation for $|\tilde{x}_3| > 1$ obtained by replacing $\tanh \rightarrow \coth$ gives dual metric and dilaton in Brinkmann coordinates

$$ds^2 = 2dudv + 2 \frac{z_3^2}{\sinh^2(u)} du^2 + dz_3^2 + dz_4^2,\tag{62}$$

$$\Phi(u) = c_1 + c_2 u + 2 \log(\sinh(u)).$$

These results are *independent of* α and valid for both S_7 and S_8 , hence we can restrict consideration to the simpler case of S_8 . Even though the adjoint representation of S_8 is not faithful, we can solve the equation (9) for coordinates of \tilde{g}

$$\begin{aligned}\tilde{x}_1 &= \tilde{h}_1 - x^2 \tilde{h}_2 - x^3 \tilde{h}_3, \\ \tilde{x}_2 &= e^{x^1} \tilde{h}_2, \quad \tilde{x}_3 = e^{x^1} \tilde{h}_3, \quad \tilde{x}_4 = \tilde{h}_4.\end{aligned}\tag{63}$$

Like in the previous section, transformations (60) and (63) enable us to find the general solution of field equations of the sigma models with metrics (61) and (62).

6.1.2 Subalgebra S_{23}

$$S_{23} = \text{Span}[\mathcal{K}_1 = L_2 + M_1 - \frac{1}{2}(P_0 + P_3), \mathcal{K}_2 = L_1 - M_2 + \alpha P_1, \\ \mathcal{K}_3 = P_0 - P_3, \mathcal{K}_4 = P_2], \quad \alpha > 0.$$

The commutation relations are

$$[\mathcal{K}_1, \mathcal{K}_2] = \alpha \mathcal{K}_3 - \mathcal{K}_4, \quad [\mathcal{K}_2, \mathcal{K}_4] = -\mathcal{K}_3, \quad \alpha > 0.$$

The transformation of coordinates in the flat background is

$$t = \frac{1}{6} \left(-(x^1)^3 - 3((x^2)^2 + 1)x^1 + 6x^3 \right), \quad x = x^2\alpha + \frac{(x^1)^2}{2}, \\ z = \frac{1}{6} \left((x^1)^3 + 3((x^2)^2 - 1)x^1 - 6x^3 \right), \quad y = x^4 - x^1x^2.$$

The flat metric in the group coordinates reads

$$F_{\mu\nu}(x) = \begin{pmatrix} 0 & \alpha x^1 & 1 & -x^2 \\ \alpha x^1 & \alpha^2 + (x^1)^2 & 0 & -x^1 \\ 1 & 0 & 0 & 0 \\ -x^2 & -x^1 & 0 & 1 \end{pmatrix},$$

and the dual background is given by

$$\tilde{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{1}{\alpha^2 + \tilde{x}_3^2} & \frac{\tilde{x}_4 - \alpha \tilde{x}_3}{\alpha^2 + \tilde{x}_3^2} & -\frac{\tilde{x}_3}{\alpha^2 + \tilde{x}_3^2} \\ 1 & \frac{\alpha \tilde{x}_3 - \tilde{x}_4}{\alpha^2 + \tilde{x}_3^2} & -\frac{(\tilde{x}_4 - \alpha \tilde{x}_3)^2}{\alpha^2 + \tilde{x}_3^2} & \frac{\tilde{x}_3(\tilde{x}_4 - \alpha \tilde{x}_3)}{\alpha^2 + \tilde{x}_3^2} \\ 0 & \frac{\tilde{x}_3}{\alpha^2 + \tilde{x}_3^2} & \frac{\tilde{x}_3(\tilde{x}_4 - \alpha \tilde{x}_3)}{\alpha^2 + \tilde{x}_3^2} & \frac{\alpha^2}{\alpha^2 + \tilde{x}_3^2} \end{pmatrix}.$$

The dual metric in Brinkmann coordinates

$$\tilde{x}_1 = \frac{1}{24(1+u^2)^{3/2}\alpha} \left(-12(-4+u^2)(1+u^2)^2\alpha^2 z_3 - 12u\sqrt{1+u^2}(2+u^2)z_3^2 \right. \\ \left. + \sqrt{1+u^2}((1+u^2)(24v+u(-48+28u^2-3u^4)\alpha^4) - 12uz_4^2) \right), \\ \tilde{x}_2 = \sqrt{1+u^2}\alpha z_4, \quad \tilde{x}_3 = u\alpha, \quad \tilde{x}_4 = \frac{1}{2}u(-4+u^2)\alpha^2 + \sqrt{1+u^2}z_3$$

then has the form

$$ds^2 = 2dudv + \frac{(2u^2 - 1)z_4^2 - 3z_3^2}{(u^2 + 1)^2} du^2 + dz_3^2 + dz_4^2,$$

while the torsion and the dilaton are

$$H = \frac{2}{1 + u^2} du \wedge dz_3 \wedge dz_4, \quad \Phi(u) = c_1 + c_2 u + \log(1 + u^2).$$

To find the general solution of field equations of the dual sigma model, we have to express the coordinates \tilde{x}_μ in terms of x^ν and \tilde{h}_k . We get

$$\begin{aligned} \tilde{x}_1 &= \tilde{h}_1 + x^2(\alpha \tilde{h}_3 - \frac{1}{2}x^2\tilde{h}_3 - \tilde{h}_4), & \tilde{x}_3 &= \tilde{h}_3, \\ \tilde{x}_2 &= \tilde{h}_2 - x^1(\alpha \tilde{h}_3 - x^2\tilde{h}_3 - \tilde{h}_4), & \tilde{x}_4 &= x^2\tilde{h}_3 + \tilde{h}_4. \end{aligned}$$

6.1.3 Subalgebra S_{25}

$$S_{25} = \text{Span}[\mathcal{K}_1 = L_2 + M_1 - \epsilon P_2, \mathcal{K}_2 = P_0 + P_3, \mathcal{K}_3 = P_1, \mathcal{K}_4 = P_0 - P_3], \quad \epsilon = \pm 1.$$

The commutation relations are

$$[\mathcal{K}_1, \mathcal{K}_2] = 2\mathcal{K}_3, \quad [\mathcal{K}_1, \mathcal{K}_3] = \mathcal{K}_4.$$

The transformation of coordinates in the flat background

$$t = x^2 + x^4, \quad x = x^3, \quad y = -\epsilon x^1, \quad z = x^2 - x^4,$$

yields the flat metric in the group coordinates

$$F_{\mu\nu}(x) = \begin{pmatrix} \epsilon^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix}.$$

The dual background is given by

$$\tilde{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix} \frac{1}{\tilde{x}_4^2 + \epsilon^2} & 0 & \frac{\tilde{x}_4}{\tilde{x}_4^2 + \epsilon^2} & -\frac{\tilde{x}_3}{\tilde{x}_4^2 + \epsilon^2} \\ 0 & 0 & 0 & -\frac{1}{2} \\ -\frac{\tilde{x}_4}{\tilde{x}_4^2 + \epsilon^2} & 0 & \frac{\epsilon^2}{\tilde{x}_4^2 + \epsilon^2} & \frac{\tilde{x}_3 \tilde{x}_4}{\tilde{x}_4^2 + \epsilon^2} \\ \frac{\tilde{x}_3}{\tilde{x}_4^2 + \epsilon^2} & -\frac{1}{2} & \frac{\tilde{x}_3 \tilde{x}_4}{\tilde{x}_4^2 + \epsilon^2} & -\frac{\tilde{x}_3^2}{\tilde{x}_4^2 + \epsilon^2} \end{pmatrix}.$$

The transformation to Brinkmann coordinates

$$\begin{aligned}\tilde{x}_1 &= \epsilon \sqrt{u^2 + 1} z_4, & \tilde{x}_3 &= \sqrt{u^2 + 1} z_3, \\ \tilde{x}_2 &= \frac{1}{\epsilon(u^2 + 1)} [u (u^2 + 2) z_3^2 + u z_4^2] - 2\epsilon v, & \tilde{x}_4 &= \epsilon u.\end{aligned}$$

gives the dual metric

$$ds^2 = 2dudv + \frac{(2u^2 - 1)z_4^2 - 3z_3^2}{(u^2 + 1)^2} du^2 + dz_3^2 + dz_4^2.$$

The torsion and dilaton then read

$$H = -\frac{2}{1 + u^2} du \wedge dz_3 \wedge dz_4, \quad \Phi(u) = c_1 + c_2 u + \log(1 + u^2).$$

To find the general solution of field equations of the dual sigma model, we have to express the coordinates \tilde{x}_μ in terms of x^ν and \tilde{h}_k . We get

$$\begin{aligned}\tilde{x}_1 &= \tilde{h}_1 + 2x^2 \tilde{h}_3 + x^3 \tilde{h}_4, & \tilde{x}_3 &= \tilde{h}_3 - x^1 \tilde{h}_4, \\ \tilde{x}_2 &= \tilde{h}_2 - x^1 (2\tilde{h}_3 - x^1 \tilde{h}_4), & \tilde{x}_4 &= \tilde{h}_4.\end{aligned}$$

6.1.4 Subalgebras S_{26} , S_{27}

The subalgebras S_{26}, S_{27} differ once again only in the value of the parameter α : it is positive for S_{26} , while $\alpha = 0$ for S_{27} [17].

$$\begin{aligned}S_{26} &= \text{Span}[\mathcal{K}_1 = M_3 + \alpha P_1, \mathcal{K}_2 = L_2 + M_1, \mathcal{K}_3 = L_1 - M_2, \mathcal{K}_4 = P_0 - P_3], \\ S_{27} &= \text{Span}[\mathcal{K}_1 = M_3, \mathcal{K}_2 = L_2 + M_1, \mathcal{K}_3 = L_1 - M_2, \mathcal{K}_4 = P_0 - P_3].\end{aligned}$$

Their commutation relations are

$$[\mathcal{K}_1, \mathcal{K}_2] = -\mathcal{K}_2 - \alpha \mathcal{K}_4, \quad [\mathcal{K}_1, \mathcal{K}_3] = -\mathcal{K}_3, \quad [\mathcal{K}_1, \mathcal{K}_4] = -\mathcal{K}_4.$$

The transformation of coordinates in the flat background

$$\begin{aligned}t &= -\alpha x^1 x^2 + \frac{1}{2} e^{-x^1} ((x^2)^2 + (x^3)^2 + 1) + x^4, & x &= \alpha x^1 - e^{-x^1} x^2, \\ z &= \alpha x^1 x^2 - \frac{1}{2} e^{-x^1} ((x^2)^2 + (x^3)^2 - 1) - x^4, & y &= e^{-x^1} x^3\end{aligned}$$

gives the flat metric in the group coordinates

$$F_{\mu\nu}(x) = \begin{pmatrix} \alpha^2 & -e^{-x^1}\alpha(x^1+1) & 0 & e^{-x^1} \\ -e^{-x^1}\alpha(x^1+1) & e^{-2x^1} & 0 & 0 \\ 0 & 0 & e^{-2x^1} & 0 \\ e^{-x^1} & 0 & 0 & 0 \end{pmatrix}.$$

In the dual background

$$\tilde{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{1-\tilde{x}_4} \\ 0 & 1 & 0 & \frac{\tilde{x}_4\alpha+\alpha+\tilde{x}_2}{1-\tilde{x}_4} \\ 0 & 0 & 1 & \frac{\tilde{x}_3}{1-\tilde{x}_4} \\ \frac{1}{\tilde{x}_4+1} & \frac{-\tilde{x}_4\alpha+\alpha-\tilde{x}_2}{\tilde{x}_4+1} & -\frac{\tilde{x}_3}{\tilde{x}_4+1} & \frac{\tilde{x}_2^2+2\alpha\tilde{x}_4\tilde{x}_2+\tilde{x}_3^2+\alpha^2\tilde{x}_4^2}{\tilde{x}_4^2-1} \end{pmatrix}$$

the torsion vanishes.

The transformation to Brinkmann coordinates

$$\begin{aligned} \tilde{x}_1 &= -v + \frac{1}{8} \left(-4u\alpha^2 + \tanh(u) (4(z_3^2 + z_4^2 + \alpha^2) + \right. \\ &\quad \left. \alpha^2 \log(1 - \tanh^2(u)) (\log(1 - \tanh^2(u)) + 4)) - \right. \\ &\quad \left. 4z_3\alpha (\log(1 - \tanh^2(u)) + 2) \right), \\ \tilde{x}_2 &= z_3 - \frac{1}{2}\alpha \tanh(u) \log(1 - \tanh^2(u)), \\ \tilde{x}_3 &= z_4, \\ \tilde{x}_4 &= -\tanh(u), \end{aligned}$$

for $|\tilde{x}_1| < 1$, brings the dual metric and dilaton to forms independent of α

$$ds^2 = 2dudv - 2 \frac{z_3^2 + z_4^2}{\cosh^2(u)} du^2 + dz_3^2 + dz_4^2,$$

$$\Phi(u) = c_1 + c_2 u + 4 \log(\cosh(u)).$$

A similar transformation (see Sec. 5.1) gives the dual metric and dilaton for $|\tilde{x}_1| > 1$ in Brinkmann coordinates

$$ds^2 = 2dudv + 2 \frac{z_3^2 + z_4^2}{\sinh^2(u)} du^2 + dz_3^2 + dz_4^2,$$

$$\Phi(u) = c_1 + c_2 u + 4 \log(\sinh(u)).$$

The solution of the field equations of the dual sigma models was found in Sec. 5.1.

6.1.5 Subalgebra S_{28}

$$S_{28} = \text{Span}[\mathcal{K}_1 = L_3 - \beta M_3, \mathcal{K}_2 = L_2 + M_1, \mathcal{K}_3 = L_1 - M_2, \mathcal{K}_4 = P_0 - P_3], \quad \beta \neq 0.$$

The commutation relations are

$$[\mathcal{K}_1, \mathcal{K}_2] = \beta \mathcal{K}_2 - \mathcal{K}_3, \quad [\mathcal{K}_1, \mathcal{K}_3] = \mathcal{K}_2 + \beta \mathcal{K}_3, \quad [\mathcal{K}_1, \mathcal{K}_4] = \beta \mathcal{K}_4, \quad \beta \neq 0.$$

The transformation of coordinates in the flat background

$$\begin{aligned} t &= \frac{1}{2} ((x^2)^2 + (x^3)^2 + 1) e^{x^1 \beta} + x^4, & x &= x^2 (-e^{x^1 \beta}), \\ z &= -\frac{1}{2} ((x^2)^2 + (x^3)^2 - 1) e^{x^1 \beta} - x^4, & y &= x^3 e^{x^1 \beta}, \end{aligned}$$

gives the flat metric in the group coordinates

$$F_{\mu\nu}(x) = \begin{pmatrix} 0 & 0 & 0 & -e^{\beta x^1} \beta \\ 0 & e^{2\beta x^1} & 0 & 0 \\ 0 & 0 & e^{2\beta x^1} & 0 \\ -e^{\beta x^1} \beta & 0 & 0 & 0 \end{pmatrix}.$$

After the transformation

$$\begin{aligned} \tilde{x}_1 &= \frac{1}{2} \beta (2v - \tanh(u) (z_3^2 + z_4^2)), \\ \tilde{x}_2 &= z_3 \cos\left(\frac{\log(\cosh(u))}{\beta}\right) + z_4 \sin\left(\frac{\log(\cosh(u))}{\beta}\right), \\ \tilde{x}_3 &= z_4 \cos\left(\frac{\log(\cosh(u))}{\beta}\right) - z_3 \sin\left(\frac{\log(\cosh(u))}{\beta}\right), \\ \tilde{x}_4 &= -\tanh(u) \end{aligned}$$

of the dual background

$$\tilde{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\beta(\tilde{x}_4-1)} \\ 0 & 1 & 0 & \frac{\beta\tilde{x}_2-\tilde{x}_3}{\beta-\beta\tilde{x}_4} \\ 0 & 0 & 1 & \frac{\tilde{x}_2+\beta\tilde{x}_3}{\beta-\beta\tilde{x}_4} \\ -\frac{1}{\tilde{x}_4\beta+\beta} & \frac{\tilde{x}_3-\beta\tilde{x}_2}{\beta(\tilde{x}_4+1)} & -\frac{\tilde{x}_2+\beta\tilde{x}_3}{\tilde{x}_4\beta+\beta} & \frac{(\beta^2+1)(\tilde{x}_2^2+\tilde{x}_3^2)}{\beta^2(\tilde{x}_4^2-1)} \end{pmatrix}$$

the dual metric and the dilaton for $|\tilde{x}_1| < 1$ are expressed in the Brinkmann coordinates as

$$ds^2 = 2dudv - \frac{(z_3^2 + z_4^2) (1 + 2\beta^2 \text{sech}^2(u))}{\beta^2} du^2 + dz_3^2 + dz_4^2,$$

$$\Phi(u) = c_1 + c_2 u + 4 \log(\cosh(u)).$$

The dual metric and the dilaton for $|\tilde{x}_1| > 1$ in Brinkmann coordinates are

$$ds^2 = 2dudv - \frac{(z_3^2 + z_4^2)(1 - 2\beta^2 \operatorname{csch}^2(u))}{\beta^2} du^2 + dz_3^2 + dz_4^2,$$

$$\Phi(u) = c_1 + c_2 u + 4 \log(\sinh(u)).$$

In both cases the torsion is of the form

$$H = -\frac{2}{\beta} du \wedge dz_3 \wedge dz_4.$$

To find the solution of the equations of the dual sigma model, we also need \tilde{x}_j, h^k expressed in terms of x^j as

$$\begin{aligned} \tilde{x}_1 &= x^2 \tilde{h}_2 \beta + x^3 \tilde{h}_3 \beta + x^4 \tilde{h}_4 \beta + \tilde{h}_1 + x^3 \tilde{h}_2 - x^2 \tilde{h}_3, \\ \tilde{x}_2 &= e^{-\beta \tilde{x}_1} (\tilde{h}_3 \sin(x^1) + \tilde{h}_2 \cos(x^1)), \\ \tilde{x}_3 &= e^{-\beta \tilde{x}_1} (-\tilde{h}_2 \sin(x^1) + \tilde{h}_3 \cos(x^1)), \\ \tilde{x}_4 &= \tilde{h}_4 e^{-\beta \tilde{x}_1}. \end{aligned}$$

6.1.6 Subalgebra S_{29}

$$S_{29} = \operatorname{Span}[\mathcal{K}_1 = L_3 - \beta M_3, \mathcal{K}_2 = P_0 - P_3, \mathcal{K}_3 = P_1, \mathcal{K}_4 = P_2], \quad \beta \neq 0.$$

The commutation relations are

$$[\mathcal{K}_1, \mathcal{K}_2] = \beta \mathcal{K}_2, \quad [\mathcal{K}_1, \mathcal{K}_3] = \mathcal{K}_4, \quad [\mathcal{K}_1, \mathcal{K}_4] = -\mathcal{K}_3, \quad \beta \neq 0.$$

The transformation of coordinates in the flat background

$$t = -\frac{1}{2} e^{x^1 \beta} + x^2, \quad x = x^3, \quad y = x^4, \quad z = -\frac{1}{2} (e^{x^1 \beta}) - x^2$$

gives the flat metric in the group coordinates

$$F_{\mu\nu}(x) = \begin{pmatrix} 0 & e^{\beta x^1} \beta & 0 & 0 \\ e^{\beta x^1} \beta & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The dual background is

$$\tilde{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix} 0 & \frac{1}{\tilde{x}_2\beta+\beta} & 0 & 0 \\ \frac{1}{\beta-\beta\tilde{x}_2} & \frac{\tilde{x}_3^2+\tilde{x}_4^2}{\beta^2(\tilde{x}_2^2-1)} & \frac{\tilde{x}_4}{\beta-\beta\tilde{x}_2} & \frac{\tilde{x}_3}{\beta(\tilde{x}_2-1)} \\ 0 & -\frac{\tilde{x}_4}{\tilde{x}_2\beta+\beta} & 1 & 0 \\ 0 & \frac{\tilde{x}_3}{\tilde{x}_2\beta+\beta} & 0 & 1 \end{pmatrix}.$$

The dual metric, dilaton and torsion in Brinkmann coordinates are the same as in section 5.2

$$ds^2 = 2dudv - (z_3^2 + z_4^2) du^2 + dz_3^2 + dz_4^2,$$

$$\Phi(u) = c_1 + c_2 u, \quad H = -2 du \wedge dz_3 \wedge dz_4.$$

To find the solution of the equations of motion of the dual sigma model, we also need \tilde{x}_j, h^k expressed in terms of x^j as

$$\begin{aligned} \tilde{x}_1 &= x^2 \tilde{h}_2 \beta + \tilde{h}_1 - x^4 \tilde{h}_3 + x^3 \tilde{h}_4, & \tilde{x}_3 &= \tilde{h}_3 \cos(x^1) - \tilde{h}_4 \sin(x^1), \\ \tilde{x}_2 &= \tilde{h}_2 e^{x^1(-\beta)}, & \tilde{x}_4 &= \tilde{h}_3 \sin(x^1) + \tilde{h}_4 \cos(x^1). \end{aligned}$$

6.1.7 Subalgebras S_{31}, S_{33}

The subalgebras

$$\begin{aligned} S_{31} &= \text{Span}[\mathcal{K}_1 = M_3, \mathcal{K}_2 = P_1 + \beta P_2, \mathcal{K}_3 = P_0 - P_3, \mathcal{K}_4 = L_2 + M_1], \\ S_{33} &= \text{Span}[\mathcal{K}_1 = M_3 + \alpha P_2, \mathcal{K}_2 = P_1 + \beta P_2, \mathcal{K}_3 = P_0 - P_3, \mathcal{K}_4 = L_2 + M_1], \end{aligned}$$

differ only in the value of the parameter α : it is positive for S_{33} , while $\alpha = 0$ for S_{31} . In both cases $\beta \neq 0$. The subalgebras are isomorphic even though they are not equivalent under conjugacy through proper orthochronous Poincaré transformations. Their commutation relations are

$$[\mathcal{K}_1, \mathcal{K}_3] = -\mathcal{K}_3, \quad [\mathcal{K}_1, \mathcal{K}_4] = -\mathcal{K}_4, \quad [\mathcal{K}_2, \mathcal{K}_4] = -\mathcal{K}_3.$$

The transformation of coordinates in the flat background

$$\begin{aligned} t &= x^3 - x^2 x^4 - \frac{1}{2} e^{-x^1} ((x^4)^2 + 1), & x &= x^2 + e^{-x^1} x^4, \\ z &= -x^3 + x^2 x^4 + \frac{1}{2} e^{-x^1} ((x^4)^2 - 1), & y &= x^1 \alpha + x^2 \beta, \end{aligned}$$

gives the flat metric in the group coordinates as

$$F_{\mu\nu}(x) = \begin{pmatrix} \alpha^2 & \alpha\beta & -e^{-x^1} & e^{-x^1}x^2 \\ \alpha\beta & \beta^2 + 1 & 0 & e^{-x^1} \\ -e^{-x^1} & 0 & 0 & 0 \\ e^{-x^1}x^2 & e^{-x^1} & 0 & e^{-2x^1} \end{pmatrix}.$$

For the dual background

$$\tilde{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix} 0 & 0 & -\frac{1}{\tilde{x}_3+1} & 0 \\ 0 & \frac{1}{\beta^2+\tilde{x}_3^2} & \frac{\alpha\beta+(\tilde{x}_3+1)\tilde{x}_4}{(\tilde{x}_3+1)(\beta^2+\tilde{x}_3^2)} & -\frac{\tilde{x}_3+1}{\beta^2+\tilde{x}_3^2} \\ \frac{1}{\tilde{x}_3-1} & \frac{-\alpha\beta-\tilde{x}_3\tilde{x}_4+\tilde{x}_4}{(\tilde{x}_3-1)(\beta^2+\tilde{x}_3^2)} & \frac{\alpha^2\tilde{x}_3^2-2\alpha\beta\tilde{x}_4\tilde{x}_3+(\beta^2+1)\tilde{x}_4^2}{(\tilde{x}_3-1)(\beta^2+\tilde{x}_3^2)} & \frac{\alpha\beta(\tilde{x}_3+1)-(\beta^2+1)\tilde{x}_4}{(\tilde{x}_3-1)(\beta^2+\tilde{x}_3^2)} \\ 0 & \frac{\tilde{x}_3-1}{\beta^2+\tilde{x}_3^2} & \frac{\alpha\beta(\tilde{x}_3-1)-(\beta^2+1)\tilde{x}_4}{(\tilde{x}_3+1)(\beta^2+\tilde{x}_3^2)} & \frac{\beta^2+1}{\beta^2+\tilde{x}_3^2} \end{pmatrix},$$

we can find a rather complicated coordinate transformation that enables us to eliminate the α -dependence of the background. The dual metric, torsion and dilaton for $|x_3| < 1$ in Brinkmann coordinates are

$$\begin{aligned} ds^2 &= 2dudv + \left[z_3^2 \frac{\text{sech}^4(u) (2(\beta^2 + 1) \sinh^2(u) - \beta^2)}{(\tanh^2(u) + \beta^2)^2} \right. \\ &\quad \left. - z_4^2 \frac{\beta^2 \text{sech}^4(u) (2(\beta^2 + 1) \cosh^2(u) + 1)}{(\tanh^2(u) + \beta^2)^2} \right] du^2 + dz_3^2 + dz_4^2, \\ H &= \frac{2\beta}{\beta^2 \cosh^2(u) + \sinh^2(u)} du \wedge dz_3 \wedge dz_4, \\ \Phi(u) &= c_1 + c_2 u + \log((\beta^2 + 1) \cosh(2u) + \beta^2 - 1). \end{aligned}$$

The dual metric, torsion and dilaton for $|x_3| > 1$ in Brinkmann coordinates are

$$\begin{aligned} ds^2 &= 2dudv + \left[z_4^2 \frac{\beta^2 \text{csch}^4(u) (2(\beta^2 + 1) \sinh^2(u) - 1)}{(\coth^2(u) + \beta^2)^2} \right. \\ &\quad \left. - z_3^2 \frac{\text{csch}^4(u) (2(\beta^2 + 1) \cosh^2(u) + \beta^2)}{(\coth^2(u) + \beta^2)^2} \right] du^2 + dz_3^2 + dz_4^2, \\ H &= -\frac{2\beta}{\cosh^2(u) + \beta^2 \sinh^2(u)} du \wedge dz_3 \wedge dz_4, \end{aligned}$$

$$\Phi(u) = c_1 + c_2 u + \log((\beta^2 + 1) \cosh(2u) - \beta^2 + 1).$$

To find the general solution of field equations of dual sigma model, it is sufficient to express the coordinates \tilde{x}_μ in terms of x^ν and \tilde{h}_k for $\alpha = 0$. We get

$$\begin{aligned}\tilde{x}_1 &= \tilde{h}_1 - x^3 \tilde{h}_3 - x^4 \tilde{h}_4, & \tilde{x}_2 &= \tilde{h}_2 - x^4 \tilde{h}_3, \\ \tilde{x}_3 &= e^{x^1} \tilde{h}_3, & \tilde{x}_4 &= e^{x^1} (x^2 \tilde{h}_3 + \tilde{h}_4).\end{aligned}$$

6.2 Diagonalizable metrics with nontrivial scalar curvature

6.2.1 Subalgebra S_{11}

$$S_{11} = \text{Span}[\mathcal{K}_1 = M_3 + \alpha P_2, \mathcal{K}_2 = P_0, \mathcal{K}_3 = P_3, \mathcal{K}_4 = P_1], \quad \alpha > 0.$$

The commutation relations are

$$[\mathcal{K}_1, \mathcal{K}_2] = \mathcal{K}_3, \quad [\mathcal{K}_1, \mathcal{K}_3] = \mathcal{K}_2.$$

The flat metric in the group coordinates

$$x^1 = \frac{y}{\alpha}, \quad x^2 = t, \quad x^3 = z, \quad x^4 = x$$

reads

$$F_{\mu\nu} = \begin{pmatrix} \alpha^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The dual background

$$\tilde{F}_{\mu\nu} = \begin{pmatrix} \frac{1}{\alpha^2 + \tilde{x}_2^2 - \tilde{x}_3^2} & -\frac{\tilde{x}_3}{\alpha^2 + \tilde{x}_2^2 - \tilde{x}_3^2} & \frac{\tilde{x}_2}{\alpha^2 + \tilde{x}_2^2 - \tilde{x}_3^2} & 0 \\ \frac{\tilde{x}_3}{\alpha^2 + \tilde{x}_2^2 - \tilde{x}_3^2} & -\frac{\alpha^2 + \tilde{x}_2^2}{\alpha^2 + \tilde{x}_2^2 - \tilde{x}_3^2} & \frac{\tilde{x}_2 \tilde{x}_3}{\alpha^2 + \tilde{x}_2^2 - \tilde{x}_3^2} & 0 \\ -\frac{\tilde{x}_2}{\alpha^2 + \tilde{x}_2^2 - \tilde{x}_3^2} & \frac{\tilde{x}_2 \tilde{x}_3}{\alpha^2 + \tilde{x}_2^2 - \tilde{x}_3^2} & \frac{\alpha^2 - \tilde{x}_3^2}{\alpha^2 + \tilde{x}_2^2 - \tilde{x}_3^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

gives a metric with nonvanishing scalar curvature

$$\tilde{R} = \frac{2(2\tilde{x}_2^2 - 2\tilde{x}_3^2 - 5\alpha^2)}{(\tilde{x}_2^2 - \tilde{x}_3^2 + \alpha^2)^2},$$

so that it cannot be transformed to the pp-wave form. On the other hand, the metric of this background can be diagonalized to the time-dependent form

$$ds^2 = -dy_1^2 + dy_2^2 + \frac{y_1^2 \alpha^2}{y_1^2 + \alpha^2} dy_3^2 + \frac{1}{y_1^2 + \alpha^2} dy_4^2, \quad (64)$$

via

$$\tilde{x}_1 = y_4, \quad \tilde{x}_2 = y_1 \cosh y_3, \quad \tilde{x}_3 = y_1 \sinh y_3, \quad \tilde{x}_4 = y_2.$$

The torsion then acquires the form

$$H = -\frac{2y_1 \alpha^2}{(y_1^2 + \alpha^2)^2} dy_1 \wedge dy_3 \wedge dy_4,$$

and the dilaton satisfying (19)–(21) is

$$\Phi = \log(y_1^2 + \alpha^2) + \text{const.}$$

To find the solution of the equations of this dual sigma model, we need the above transformation between y_j and \tilde{x}_j , and also \tilde{x}_j expressed in terms of x^j, \tilde{h}_k .

$$\begin{aligned} \tilde{x}_1 &= \tilde{h}_1 + x^2 \tilde{h}_3 + x^3 \tilde{h}_2, & \tilde{x}_2 &= \tilde{h}_2 \cosh x^1 - \tilde{h}_3 \sinh x^1, \\ \tilde{x}_4 &= \tilde{h}_4, & \tilde{x}_3 &= \tilde{h}_3 \cosh x^1 - \tilde{h}_2 \sinh x^1. \end{aligned}$$

6.2.2 Subalgebra S_{18}

$$S_{18} = \text{Span}[\mathcal{K}_1 = L_3 + \alpha P_0, \mathcal{K}_2 = P_1, \mathcal{K}_3 = P_2, \mathcal{K}_4 = P_3], \quad \alpha > 0.$$

The commutation relations are the same as for S_{17} and S_{19} ,

$$[\mathcal{K}_1, \mathcal{K}_2] = \mathcal{K}_3, \quad [\mathcal{K}_1, \mathcal{K}_3] = -\mathcal{K}_2.$$

The flat metric in the group coordinates

$$x^1 = \frac{t}{\alpha}, \quad x^2 = x, \quad x^3 = y, \quad x^4 = z$$

reads

$$F_{\mu\nu} = \begin{pmatrix} -\alpha^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The dual background

$$\tilde{F}_{\mu\nu} = \begin{pmatrix} \frac{1}{-\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2} & \frac{\tilde{x}_3}{-\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2} & -\frac{\tilde{x}_2}{-\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2} & 0 \\ -\frac{\tilde{x}_3}{-\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2} & \frac{\tilde{x}_2^2 - \alpha^2}{-\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2} & \frac{\tilde{x}_2 \tilde{x}_3}{-\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2} & 0 \\ \frac{\tilde{x}_2}{-\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2} & \frac{\tilde{x}_2 \tilde{x}_3}{-\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2} & \frac{\tilde{x}_3^2 - \alpha^2}{-\alpha^2 + \tilde{x}_2^2 + \tilde{x}_3^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

gives a metric with nonvanishing scalar curvature

$$\tilde{R} = -\frac{10\alpha^2 + 4(\tilde{x}_2^2 + \tilde{x}_3^2)}{(\tilde{x}_2^2 + \tilde{x}_3^2 - \alpha^2)^2},$$

so it cannot be transformed to the pp-wave form. On the other hand, the metric of this background can be diagonalized to the form

$$ds^2 = \frac{1}{y_3^2 - \alpha^2} dy_1^2 + \frac{y_3^2 \alpha^2}{\alpha^2 - y_3^2} dy_2^2 + dy_3^2 + dy_4^2, \quad (65)$$

by

$$\tilde{x}_1 = y_1, \quad \tilde{x}_2 = y_3 \cos y_2, \quad \tilde{x}_3 = y_3 \sin y_2, \quad \tilde{x}_4 = y_4.$$

Note the singularity on the surfaces $y_3 = \pm\alpha$. For $|y_3| < \alpha$ the time-like direction is given by the vector ∂_{y_1} , whereas for $|y_3| > \alpha$ the time-like vector is ∂_{y_2} .

The torsion acquires the form

$$H = \frac{2y_3\alpha^2}{(y_3^2 - \alpha^2)^2} dy_1 \wedge dy_2 \wedge dy_3$$

and the dilaton satisfying (19)–(21) is

$$\Phi = \log(y_3^2 - \alpha^2) + \text{const.}$$

To find the solution of the equations of this dual sigma model, we need the above transformation between y_j and \tilde{x}_j , and also \tilde{x}_j expressed in terms of x^j, \tilde{h}_k . As the commutation relations are the same as for S_{17} and S_{19} , we get

$$\begin{aligned} \tilde{x}_1 &= \tilde{h}_1 + x^2 \tilde{h}_3 - x^3 \tilde{h}_2, & \tilde{x}_2 &= \tilde{h}_2 \cos x^1 - \tilde{h}_3 \sin x^1, \\ \tilde{x}_4 &= \tilde{h}_4, & \tilde{x}_3 &= \tilde{h}_2 \sin x^1 + \tilde{h}_3 \cos x^1. \end{aligned}$$

7 Conclusion

We have classified all atomic non-Abelian duals of the four-dimensional flat spacetime with respect to four-dimensional subgroups of the Poincaré group. As a result, we have obtained 14 different types of exactly solvable sigma models in the four-dimensional curved backgrounds. Due to the non-Abelian T-duality, one can find general solutions of the classical field equations for all of these dual models in terms of d'Alembert solutions of the wave equation. The method of obtaining the solutions is described in section 3 and examples are given in section 5 and 6. One-loop beta equations for all of the dual backgrounds yield simple ordinary differential equations for dilatons. Their solutions are given in sections 5 and 6.

Eleven of the dual backgrounds are plane-parallel waves whose metrics can be brought to the Brinkmann form

$$ds^2 = 2dudv - [K_3(u)z_3^2 + K_4(u)z_4^2]du^2 + dz_3^2 + dz_4^2.$$

The torsion then is

$$H = dB = H(u) du \wedge dz_3 \wedge dz_4.$$

Depending on the chosen subgroup, functions $K_3(u)$, $K_4(u)$, $H(u)$ acquire various forms, as follows:

$$K_3(u) = K_4(u) = 1, \quad H(u) = -2, \quad (66)$$

$$K_3(u) = \frac{3}{(u^2 + 1)^2}, \quad K_4(u) = -\frac{(2u^2 - 1)}{(u^2 + 1)^2}, \quad H(u) = \pm \frac{2}{u^2 + 1}, \quad (67)$$

$$K_3(u) = 2 \operatorname{sech}^2(u), \quad K_4(u) = 2 \delta \operatorname{sech}^2(u), \quad \delta = 0, 1, \quad H(u) = 0, \quad (68)$$

$$K_3(u) = -2 \operatorname{csch}^2(u), \quad K_4(u) = -2 \delta \operatorname{csch}^2(u), \quad \delta = 0, 1, \quad H(u) = 0, \quad (69)$$

$$K_3(u) = K_4(u) = \frac{(1 + 2\beta^2 \operatorname{sech}^2(u))}{\beta^2}, \quad H(u) = -\frac{2}{\beta}, \quad (70)$$

$$K_3(u) = K_4(u) = \frac{(1 - 2\beta^2 \operatorname{csch}^2(u))}{\beta^2}, \quad H(u) = -\frac{2}{\beta}, \quad (71)$$

$$\begin{aligned}
K_3(u) &= -\frac{\operatorname{sech}^4(u) (2(\beta^2 + 1) \sinh^2(u) - \beta^2)}{(\tanh^2(u) + \beta^2)^2}, \\
K_4(u) &= \frac{\beta^2 \operatorname{sech}^4(u) (2(\beta^2 + 1) \cosh^2(u) + 1)}{(\tanh^2(u) + \beta^2)^2}, \\
H(u) &= \frac{2\beta}{\beta^2 \cosh^2(u) + \sinh^2(u)},
\end{aligned} \tag{72}$$

$$\begin{aligned}
K_3(u) &= \frac{\operatorname{csch}^4(u) (2(\beta^2 + 1) \cosh^2(u) + \beta^2)}{(\coth^2(u) + \beta^2)^2}, \\
K_4(u) &= -\frac{\beta^2 \operatorname{csch}^4(u) (2(\beta^2 + 1) \sinh^2(u) - 1)}{(\coth^2(u) + \beta^2)^2}, \\
H(u) &= -\frac{2\beta}{\cosh^2(u) + \beta^2 \sinh^2(u)},
\end{aligned} \tag{73}$$

where $\beta \in \mathbb{R} \setminus \{0\}$.

Even though the B-fields obtained by T-duality are usually not of the form $B = B_i(u) du \wedge dz_i$, they are gauge equivalent to

$$B' = H(u) du \wedge (z_3 dz_4 - z_4 dz_3),$$

and the corresponding sigma models are exactly conformal [3].

Except for (70), (71), these pp-wave backgrounds can be transformed to the gauged WZW background forms (1) by the standard transformation from Brinkmann to Rosen coordinates [19]. In most of the transformed backgrounds the function g_1 acquires the form $g_1(u) = 1$ and the function g_2 acquires the form of one of the functions (2), but some other combinations of functions (g_1, g_2) also arise, namely $(u^{-2}, \tanh^2 u)$, $(u^{-2}, \coth^2 u)$, $(\tanh^2 u, \tanh^2 u)$ and $(\coth^2 u, \coth^2 u)$.

Consequently, the pp-waves of the form (1) are duals of the flat metric not only for $g_1(u) = 1$ and $g_2(u) = u^2$, as mentioned in section 1, but also for many other combinations of functions g_1, g_2 from the set (2).

It is a remarkable fact that duals with respect to subgroups corresponding to non-isomorphic algebras may lead to the same backgrounds (up to coordinate transformation). These are the cases of metric (66) produced by

subalgebras S_{17} and S_{29} , and also metrics (68), (69) obtained from S_7 and S_8 for $\delta = 0$, and S_{26} and S_{27} for $\delta = 1$. The metric (66) is apparently a homogeneous exactly solvable model with nontrivial constant torsion. On the other hand, isomorphic (but not equivalent under proper orthochronous Poincaré transformations) algebras S_{23} and S_{25} give the same metric, namely (67), but opposite torsions. Isomorphic algebras S_{31} and S_{33} give the same metrics and torsions (72), (73).

We also get, besides the pp-waves, dual metrics with nonvanishing scalar curvature and torsion:

$$ds^2 = -dy_1^2 + dy_2^2 + \frac{y_1^2}{y_1^2 + \alpha^2} dy_3^2 + \frac{1}{y_1^2 + \alpha^2} dy_4^2, \quad (74)$$

$$H = -\frac{2y_1\alpha}{(y_1^2 + \alpha^2)^2} dy_1 \wedge dy_3 \wedge dy_4,$$

$$ds^2 = \frac{1}{y_3^2 - \alpha^2} dy_1^2 + \frac{y_3^2}{\alpha^2 - y_3^2} dy_2^2 + dy_3^2 + dy_4^2, \quad (75)$$

$$H = \frac{2y_3\alpha}{(y_3^2 - \alpha^2)^2} dy_1 \wedge dy_2 \wedge dy_3$$

$$ds^2 = -dy_1^2 + dy_2^2 + \frac{y_2^2}{y_2^2 + \alpha^2} dy_3^2 + \frac{1}{y_2^2 + \alpha^2} dy_4^2 \quad (76)$$

$$H = \frac{2y_2\alpha}{(y_2^2 + \alpha^2)^2} dy_2 \wedge dy_3 \wedge dy_4$$

They are obtained as non-Abelian duals with respect to S_{11}, S_{18}, S_{19} . Note that isomorphic (but not equivalent under proper orthochronous Poincaré transformation) subalgebras S_{17} (resp., S_{18}, S_{19}) lead to backgrounds with vanishing (resp., nonvanishing) curvature.

The metrics (74)–(76) remind us of black hole [10] and cosmological backgrounds [11] rewritten in [2] into diagonal forms depending again on particular functions g_1, g_2 . The difference from (74)–(76) lies in these functions.

Appendix: Poincaré subalgebras

We summarize the four-dimensional Poincaré subalgebras that act freely and transitively on the flat manifold. The numbering of the subalgebras follows from the order introduced in [17], Table IV.

$$\begin{aligned}
S_1 &= \text{Span}[P_0, P_1, P_2, P_3], \\
S_2 &= \text{Span}[M_3, P_0 - P_3, P_1, P_2], \\
S_6 &= \text{Span}[L_2 + M_1 - \frac{1}{2}(P_0 + P_3), P_1, P_0 - P_3, P_2], \\
S_7 &= \text{Span}[2M_3 + \alpha P_1, L_2 + M_1, P_0 - P_3, P_2], & \alpha > 0, \\
S_8 &= \text{Span}[M_3, L_2 + M_1, P_0 - P_3, P_2], \\
S_{11} &= \text{Span}[M_3 + \alpha P_2, P_0, P_3, P_1], & \alpha > 0, \\
S_{17} &= \text{Span}[L_3 + \epsilon(P_0 + P_3), P_1, P_2, (P_0 - P_3)], & \epsilon = \pm 1 \\
S_{18} &= \text{Span}[L_3 + \alpha P_0, P_1, P_2, P_3], & \alpha > 0, \\
S_{19} &= \text{Span}[L_3 + \alpha P_3, P_1, P_2, P_0], & \alpha \neq 0, \\
S_{23} &= \text{Span}[L_2 + M_1 - \frac{1}{2}(P_0 + P_3), L_1 - M_2 + \alpha P_1, P_0 - P_3, P_2], & \alpha \neq 0, \\
S_{25} &= \text{Span}[L_2 + M_1 - \epsilon P_2, P_0 + P_3, P_1, P_0 - P_3], & \epsilon = \pm 1 \\
S_{26} &= \text{Span}[M_3 + \alpha P_1, L_2 + M_1, L_1 - M_2, P_0 - P_3], & \alpha > 0, \\
S_{27} &= \text{Span}[M_3, L_2 + M_1, L_1 - M_2, P_0 - P_3], \\
S_{28} &= \text{Span}[L_3 - \beta M_3, L_2 + M_1, L_1 - M_2, P_0 - P_3], & \beta \neq 0, \\
S_{29} &= \text{Span}[L_3 - \beta M_3, P_0 - P_3, P_1, P_2], & \beta \neq 0, \\
S_{31} &= \text{Span}[M_3, P_1 + \beta P_2, P_0 - P_3, L_2 + M_1], & \beta \neq 0, \\
S_{33} &= \text{Span}[M_3 + \alpha P_2, P_1 + \beta P_2, P_0 - P_3, L_2 + M_1], & \alpha > 0, \beta \neq 0.
\end{aligned}$$

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